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# Compact $h^4$ Finite-Difference Approximations to Operators of Navier–Stokes Type

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A method of obtaining compact finite-difference approximations of $h^4$ accuracy to operators of Navier–Stokes type is considered. The basic procedure is developed for operators in one space dimension and subsequently applied to problems in more space dimensions and in time. Four illustrative numerical examples are given which indicate clearly in various cases that excellent accuracy may be obtained using the methods. Comparisons with previous results and with the results of $h^2$-accurate computations are made. © 1989 Academic Press, Inc.

## 1. INTRODUCTION

In solving problems governed by the Navier–Stokes equations we are interested in approximating spatial operators of typical form $L$ defined by the equation

$$L\phi = \phi'' - u\phi' = r, \tag{1}$$

where the prime denotes differentiation with respect to a space variable, say $x$, and the function $u$ has the significance of a velocity component. In the simple 1-dimensional model (1), $u$ and $r$ would be functions of $x$, but more general problems can be considered by a combination of equations of the form (1). For example the function $\phi(x, y)$ which satisfies the equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 - u\partial\phi/\partial x - v\partial\phi/\partial y = 0, \tag{2}$$

where $(u, v)$ are 2-dimensional velocity components, can be written as the two equations

$$\partial^2\phi/\partial x^2 - u\partial\phi/\partial x = r(x, y) = -\partial^2\phi/\partial y^2 + v\partial\phi/\partial y. \tag{3}$$



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The function $\phi(x, y)$ can be interpreted as the steady state scalar vorticity in a 2-dimensional flow of an incompressible fluid in this case. Similarly the operator $L$ occurs in the transport equation

$$\partial^2\phi/\partial x^2 - u\partial\phi/\partial x = r(x, t) = \partial\phi/\partial t \tag{4}$$

for the function $\phi(x, t)$ and there are many other examples and extensions.

The standard three-point finite-difference approximation to (1) obtained by making use of central differences at a typical grid point $x = x_0$ is

$$(1 - \tfrac{1}{2}hu_0)\,\phi_1 + (1 + \tfrac{1}{2}hu_0)\,\phi_3 - 2\phi_0 - h^2 r_0 = 0, \tag{5}$$

where $h$ is the grid size and the subscripts 1, 0, 3 denote values at the respective points $x_0 + h$, $x_0$, $x_0 - h$ in accordance with the Southwell notation [1]. The approximation (5) is $h^2$-accurate in the sense that the truncation error in approximating (1) is $O(h^2)$. In practice it is desirable that the coefficient matrix associated with a finite-difference scheme is diagonally dominant, especially if iterative methods of solution are used. For example, the Jacobi and Gauss–Seidel point iterative methods both converge for matrices which are diagonally dominant (see Varga [2] or Young [3] for theorems concerning the convergence of iterative methods applied to diagonally dominant systems). It may also be noted that one of the basic underlying reasons for the use of upwind methods is that they preserve this diagonally dominant property. The approximation (5) is diagonally dominant if $|hu_0| \leqslant 2$ at every grid point of the solution domain. However, the velocity component $u$ in (1) is usually multiplied by a Reynolds number factor (here omitted for the sake of convenience) which may be large; in this case a very small step length is required to achieve diagonal dominance.

An alternative $O(h^2)$ approximation to (1) is given by

$$\begin{aligned}(1 - \tfrac{1}{2}hu_0 + \alpha h^2 u_0^2)\,\phi_1 + (1 + \tfrac{1}{2}hu_0 + \alpha h^2 u_0^2)\,\phi_3 \\ - (2 + 2\alpha h^2 u_0^2)\,\phi_0 - h^2 r_0 = 0\end{aligned} \tag{6}$$

where $\alpha$ is an arbitrary constant. This approximation has an associated matrix which is diagonally dominant for all $h$ provided that $\alpha \geqslant \frac{1}{16}$. It was proposed by Dennis and Hudson [4] for the case $\alpha = \frac{1}{8}$ and subsequently generalized to $\alpha \geqslant \frac{1}{16}$ by Lindroos [5]. Note that (6) reduces to (5) if $\alpha = 0$. The present paper extends (6) to give a 3-point formula of $h^4$ accuracy. In this extension we put $\alpha = \frac{1}{12}$ in (6) and modify the coefficients of $\phi_0$, $\phi_1$, and $\phi_3$ to include terms involving the first and second derivatives of $u$ with respect to $x$ at $x = x_0$. The term involving $r_0$ is also modified to include terms in $r_1$ and $r_3$. When $r$ depends upon another variable, as in the case of (3) or (4), these values of $r$ can be expressed in terms of grid values of $\phi$; this leads to an $h^4$-accurate compact approximation to (2) which involves only the nine grid points centred on the point 0 of the Southwell notation [1] (indicated in Fig. 1).

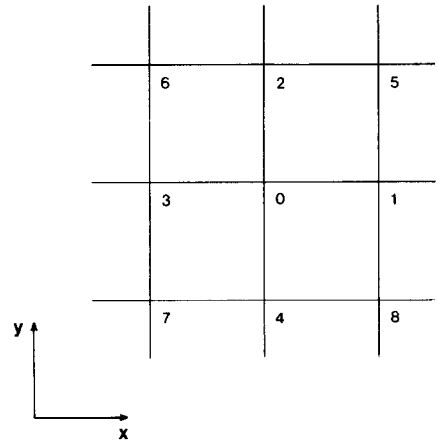


FIG. 1. The 9-point molecule.

There has been much recent interest in compact $h^4$-accurate schemes in which the first derivative $F=\phi'$ is employed as a dependent variable in addition to $\phi$. There are numerous variations, but a typical scheme would employ the approximation

$$\phi_1 - \phi_3 = \tfrac{1}{3}h(F_1 + 4F_0 + F_3) \tag{7}$$

to $\phi' = F$ along with the Numerov [6, 7] approximation

$$\phi_1 - 2\phi_0 + \phi_3 = (h^2/12)(G_1 + 10G_0 + G_3) \tag{8}$$

to (1) written in the form

$$\phi'' = uF + r = G. \tag{9}$$

Thus provided $G$ is defined, by means of suitable definitions of $u$ and $r$, then (7) and (8) define two sets of finite-difference equations with associated matrices of tri-diagonal type which may be solved to give an approximation to each of $\phi$ and $F$ over the domain of the set of grid points typified by $x = x_0$. Methods of this type have recently been reviewed by Hirsh [8] who states that it is the consideration of derivatives of $\phi$ as unknowns in the solution procedure which is the key to the higher order compact differencing procedures. However, a problem which clearly arises is that normally a boundary condition for $\phi$ or $F$, but not both, will be specified at each boundary point in the case of two-point conditions. Thus expansion procedures at the boundaries will be necessary. This will not be the case for methods involving $\phi$ alone, such as those of the present paper.

Some of the earlier compact differencing methods [9–11] did in fact eliminate completely the derivatives as dependent variables leaving ultimately an expression in terms of the variable $\phi$ alone, although it was not clear how such methods could readily be extended to problems involving more than one space variable. More

recent methods [12–14] are based in principle on expressions of the type (7)–(9) and there are compact methods involving the use of splines [15–17]. In the operator compact implicit (OCI) methods [18, 19] the whole operator in a differential equation is expressed in a compact form and there are numerous other contributions to the subject, e.g., [20, 21] through to the more recent ones, for example, the method of Leventhal [22] in which a compact implicit method of exponential type is given. Many of the contributions are discussed by Hirsh [8] and will not be considered here in detail because our object is to present the latest developments in $h^4$-accurate compact methods which involve only $\phi$.

The primary obstacle to giving a compact $h^4$-accurate approximation to (1) is the presence of the first derivative $\phi'$. Thus the Numerov approximation (8) involves the first derivative of $\phi$ but would not do so if $F$ were not present in (9). A direct extension of Numerov's method to an operator of type (1) was made by Dennis [23] by putting.

$$\chi = \phi g \qquad (x_0 - h \leqslant x \leqslant x_0 + h), \tag{10}$$

where

$$g(x) = \exp\left\{ -\frac{1}{2}\int_{x_0}^{x} u(\xi)\, d\xi \right\}. \tag{11}$$

The function $\chi$ satisfies the equation

$$\chi'' + f\chi = rg, \tag{12}$$

where

$$f = \tfrac{1}{2}u' - \tfrac{1}{4}u^2. \tag{13}$$

Because (12) does not contain $\chi'$ we can approximate it locally at $x = x_0$ using either an $h^2$-accurate method or the $h^4$-accurate Numerov method. In either case we obtain a three-point, compact approximation for $\chi$ which can be expressed in terms of the original function $\phi$ using (10). The coefficients of $\phi$ in both of the approximations depend upon $g$ in (11) and hence involve the exponentials of definite integrals of $u$ corresponding to the points $x = x_0 - h$, $x_0 + h$. The exponential forms of these coefficients were given in [23] and a slightly more general form of (1) was considered.

The $h^4$-accurate approximations with exponential coefficients were extended to the 2-dimensional equation (2) by Dennis and Hudson in two somewhat different forms in [24, 25]. In effect these results give compact approximations involving only values of $\phi$ at the nine points centred on a typical point $(x_0, y_0)$ of a 2-dimensional square grid. In the present paper it will be shown that satisfactory computational results can be obtained by employing expansion techniques in which the exponentials are expressed in powers of their arguments.

A brief review of a few aspects of the present method was given in [26]. Here, we study in detail the approximations which may be obtained to (1), (2), and (4). Results of this nature have not been given previously. It is shown that highly accurate computational results can be obtained using the methods. Examples of numerical solutions of Eqs. (1), (2), and (4) are given. The problem of heat convection in a square cavity is also considered as an example.

## 2. The Basic $H^4$-Accurate Approximation

The $h^2$-approximation to (12) using central differences gives, when expressed in terms of $\phi$ using (10),

$$g_1\phi_1 + g_3\phi_3 - (2 - h^2 f_0)\,\phi_0 - h^2 r_0 = 0 \tag{14}$$

and the $h^4$-accurate Numerov approximation gives

$$\begin{aligned}(1 + h^2 f_1/12)\,g_1\phi_1 + (1 + h^2 f_3/12)\,g_3\phi_3 - (2 - 5h^2 f_0/6)\,\phi_0 \\ -h^2(g_1 r_1 + 10 r_0 + g_3 r_3)/12 = 0.\end{aligned} \tag{15}$$

The terms $g_1$ and $g_3$ are exponential coefficients found by substituting $x = x_0 + h$ and $x = x_0 - h$, respectively, in (11). It was found in [4] that expansion of these exponential coefficients in powers of their arguments, which is valid for all arguments, followed by approximation of the definite integrals derived from (11) using a power series expansion for $u$ about $x = x_0$, gave rise to the approximation (6) with $\alpha = \frac{1}{8}$. Strictly speaking the terms in $\alpha$ in (6) should not be retained because when they are grouped together they form a term of order equal to that which has already been neglected in forming (14) from (12). They were, however, retained in [4] because the associated matrix is diagonally dominant if $\alpha = \frac{1}{8}$.

The approximation (15) is an $h^4$-accurate approximation. As it is written, the leading error neglected on the right side is $O(h^6)$ and thus it is valid in an expansion procedure to retain terms up to this order in $h$. On expansion of $g_1$ and $g_3$ to this order and identification of various terms involving $f$ and its derivatives from (13), we find several possible forms which retain the desired accuracy. For example, we can write the approximation to (15) in the form

$$c_1\phi_1 + c_3\phi_3 - c_0\phi_0 - h^2 r_0 + C_0 = 0, \tag{16}$$

where

$$c_1 = 1 - \tfrac{1}{2}hu_0 + h^2(u_0^2 - 2u_0')/12 + h^3(u_0 u_0' - u_0'')/24, \tag{17a}$$

$$c_3 = 1 + \tfrac{1}{2}hu_0 + h^2(u_0^2 - 2u_0')/12 - h^3(u_0 u_0' - u_0'')/24, \tag{17b}$$

$$c_0 = 2 + h^2(u_0^2 - 2u_0')/6, \tag{17c}$$

and $C_0$ is a term of $O(h^4)$ given by

$$C_0 = -h^2\{(1-\tfrac{1}{2}hu_0)\,r_1 + (1+\tfrac{1}{2}hu_0)\,r_3 - 2r_0\}/12. \tag{17d}$$

Details of one method of derivation are given in the Appendix.

Note that it has been possible to express the coefficients $c_n$ in (16) entirely in terms of values of $u$ and its derivatives at the point $x = x_0$. This is achieved during simplification of the expansion of (15) without any loss of accuracy, i.e., the approximation (16) with the definitions (17) is a fully $h^4$-accurate approximation to (1). Moreover, the derivatives $u_0'$ and $u_0''$ each multiply sets of terms in (16) which are, on aggregate, of order $h^4$ and hence if $u$ is given only numerically there is no further loss of order of accuracy by approximating these derivatives with the central-difference formulae

$$2hu_0' \approx u_1 - u_3, \qquad h^2u_0'' \approx u_1 - 2u_0 + u_3. \tag{18}$$

In this way (16) remains completely compact involving only values at the three points centred at $x = x_0$.

A simpler form of the $h^4$-accurate approximation can be derived from Eq. (17). If we write

$$a_1 = 1 - \tfrac{1}{4}hu_0 - h^2u_0'/12, \qquad a_3 = 1 + \tfrac{1}{4}hu_0 - h^2u_0'/12 \tag{19}$$

and utilize the fact that

$$\phi_1 + \phi_3 = 2\phi_0 + O(h^2), \tag{20}$$

we obtain

$$\begin{aligned}(a_1^2 + h^2u_0^2/48 - h^3u_0''/24)\,\phi_1 + (a_3^2 + h^2u_0^2/48 + h^3u_0''/24)\,\phi_3 \\ - (a_1^2 + a_3^2 + h^2u_0^2/24)\,\phi_0 - h^2r_0 + C_0 = 0.\end{aligned} \tag{21}$$

For ordinary differential equations, the function $r(x)$ in (1) is specified in some way as a function of $x$ and (21) then defines a matrix to determine $\phi$ subject to given two-point boundary conditions. A specific numerical example which illustrates the high accuracy obtainable is given in a later section. In cases when $\phi$ satisfies partial differential equations such as (2) or (4), Eq. (21) defines $r$ in terms of $\phi$ and then a further equation is necessary, or more than one if there are more independent variables. We consider in detail the case of Eq. (4) in the next section and Eq. (2) in the following one.

Before concluding this section, we recall that with $\alpha \geqslant \frac{1}{16}$, the original $h^2$-accurate approximation (6) has an associated matrix which is diagonally dominant under all circumstances. In contrast, the $h^4$-accurate scheme (21) has a diagonally dominant associated matrix only if

$$|h^3u_0''| \leqslant 24a^2 + \tfrac{1}{2}h^2u_0^2 \tag{22}$$

at all grid points of the solution domain, where $a$ is the least of the absolute values of $a_1$ and $a_3$. However, it is possible to rewrite the $h^4$-accurate scheme as an $h^3$-accurate approximation, having an associated matrix which is diagonally dominant under all circumstances, together with a fourth-order correction. Details are as follows.

An $h^3$-accurate approximation which always has a diagonally dominant associated matrix can be obtained from (21) using upwind methods. The terms multiplying the factor $h^3u_0''/24$ can be arranged as a backward or forward difference in $\phi$ plus a deferred correction, depending upon whether $u_0''$ is greater or less than zero. In this way (21) can be re-written as

$$(a_1^2 + h^2u_0^2/48)\,\phi_1 + (a_3^2 + h^2u_0^2/48 + h^3u_0''/12)\,\phi_3 - (a_1^2 + a_3^2 + h^2u_0^2/24 + h^3u_0''/12)\,\phi_0 - h^2r_0 + C_0 - D_0 = 0 \tag{23}$$

at grid points where $u_0'' \geqslant 0$ and as

$$(a_1^2 + h^2u_0^2/48 - h^3u_0''/12)\,\phi_1 + (a_3^2 + h^2u_0^2/48)\,\phi_3 - (a_1^2 + a_3^2 + h^2u_0^2/24 - h^3u_0''/12)\,\phi_0 - h^2r_0 + C_0 + D_0 = 0 \tag{24}$$

at grid points where $u_0'' < 0$. The deferred correction $D_0$ is defined as

$$D_0 = h^3u_0''(\phi_1 + \phi_3 - 2\phi_0)/24 \tag{25}$$

and is $O(h^5)$. If it is neglected in (23) and (24) we get an $h^3$-accurate approximation to (1) using the appropriate equation at each grid point. An approximation to the solution of (1) using this scheme can then be improved if desired by adding the deferred correction (25) to the left side of (23) or (24) and re-solving these equations iteratively until eventually the sequence of approximations to $\phi$ converges at all grid points, at which stage the $h^4$-accurate set of Eqs. (21) is satisfied. This type of deferred-correction process is well known; it was used, for example, in [27] to correct the standard first-order accurate upwind scheme to second-order central-difference accuracy by means of a sequence of computations with diagonally dominant matrices.

## 3. Equations in One Space Variable and Time

For Eq. (4) we need to approximate the equation

$$\partial\phi/\partial t = r(x, t) \tag{26}$$

using some finite-difference scheme in the time $t$ and then eliminate $r$ from (21) for an $h^4$-accurate method. The truncation error of the approximation in time will be a relevant consideration but the general principle of setting up the finite-difference

equation will be more or less the same regardless of the approximation used for (26). We shall illustrate with the Crank–Nicolson method which gives

$$\phi(x_n, t+k) - \tfrac{1}{2}kr(x_n, t+k) = \phi(x_n, t) + \tfrac{1}{2}kr(x_n, t) \tag{27}$$

as a discretization of (26) using a time step $k$. This is a $k^2$-accurate approximation in time, i.e., the truncation error on the right-hand side of (27) is $O(k^3)$. On the assumption that $\phi(x_0, t)$ and $r(x_0, t)$ have been determined at all spatial grid points, we can use (27) to eliminate $r(x_0, t+k)$ from (16), assuming that this latter equation holds at time $t+k$. There are then two main ways of dealing with the resulting equation.

In the first place, the result of this elimination can be expressed in the form

$$\begin{aligned} &c_1(t+k)\,\phi_1(t+k) + c_3(t+k)\,\phi_3(t+k) - \{2\beta + c_0(t+k)\}\,\phi_0(t+k) + C_0(t+k) \\ &\quad = -2\beta\phi_0(t) - h^2 r_0(t), \end{aligned} \tag{28a}$$

where $\beta = h^2/k$ and notation such as $\phi_n(t)$ denotes $\phi(x_n, t)$ $(n = 0, 1, 3)$. The term $C_0(t+k)$ also contains $r_0(t+k)$, $r_1(t+k)$ and $r_3(t+k)$ through (17d) which must be eliminated using (27). However, it is a term of order $h^4$ as it appears in (16) and will depend upon $\beta$ after the values of $r$ have been eliminated, so it can be left as a deferred correction in (28a). Equation (28a) can then be solved iteratively, with new approximations to $\phi_n(t+k)$ used from time to time to calculate improved estimates of $C_0(t+k)$, using (27) to calculate $r_n(t+k)$. In fact, iterative methods are frequently quite appropriate since the function $u(x, t)$ in (4) is often itself determined as the solution of another governing equation which must be solved simultaneously with the set of Eqs. (28a) in an over-all iterative procedure.

An alternative to treating $C_0(t+k)$ as a deferred correction is to eliminate completely from (28a) the terms involving $r_n(t+k)$ $(n = 0, 1, 3)$ using (27) to give an explicitly tridiagonal formulation. This gives the equivalent equation

$$\begin{aligned} &[6c_1 - \beta\{1 - \tfrac{1}{2}hu_0(t+k)\}]\phi_1(t+k) + [6c_3 - \beta\{1 + \tfrac{1}{2}hu_0(t+k)\}]\,\phi_3(t+k) \\ &\qquad - (6c_0 + 10\beta)\phi_0(t+k) \\ &\quad = -\{1 - \tfrac{1}{2}hu_0(t+k)\}[\beta\phi_1(t) + \tfrac{1}{2}h^2 r_1(t)] \\ &\qquad - \{1 + \tfrac{1}{2}hu_0(t+k)\}[\beta\phi_3(t) + \tfrac{1}{2}h^2 r_3(t)] - 10[\beta\phi_0(t) + \tfrac{1}{2}h^2 r_0(t)]. \end{aligned} \tag{28b}$$

The set of Eqs. (27) is used to calculate the quantities $r_n(t)$ $(n = 0, 1, 3)$, by replacing $t$ by $t-k$ (or using the initial data for the first time step). The right-hand side of (28b) is then known provided $u_0(t+k)$ is known and a simple tridiagonal inversion gives the solution. This may be more efficient. On the other hand, if $u(x, t)$ must itself be determined as part of the solution procedure, the direct solution procedure is probably no more efficient than an iterative procedure.

A simpler alternative to using (27) is to use the backward-difference approximation

$$r(x_n, t+k) = \{\phi(x_n, t+k) - \phi(x_n, t)\}/k \quad (n = 0, 1, 3) \tag{29}$$

to perform the elimination in (16), although this involves a higher truncation error in $k$. On the other hand we can use (29) to evaluate only the terms in (28a) which appear in the correction term $C_0(t+k)$ which is a less important term from the spatial point of view. In this latter procedure the calculation is only a little more complicated than the standard Crank–Nicolson procedure used with $h^2$-accurate spatial approximations, but the gain in spatial accuracy is considerable. There are clearly many ways to proceed, but a very effective one is to use (28a) with the coefficients $c_n(t+k)$ expressed in the form (21). The matrix for $\phi$ is inverted by an iterative procedure and the terms $C_0$ and $D_0$ (Eq. (25)) are added as deferred corrections. A numerical example of this technique is described later.

## 4. Steady State Equations in Two Space Variables

We shall now consider the $h^4$-accurate approximation to (2) in which the finite-difference representation can be expressed as the nine-point formula

$$\sum_{n=1}^{8} d_n\phi_n - d_0\phi_0 + B_0 = 0, \tag{30}$$

where the Southwell notation is again used. Equation (2) is expressed as the two equations (3). The $h^4$ approximation to the first gives (16), where the coefficients are given by (17) and the prime indicates differentiation with respect to $x$. The corresponding approximation to the second of (3) in the $y$ direction is

$$c_2\phi_2 + c_4\phi_4 - c_0^*\phi_0 + h^2 r_0 + C_0^* = 0. \tag{31}$$

Here the coefficients $c_2$, $c_4$ and $c_0^*$ are given respectively by the right-hand sides of (17a)–(17c) but with $v$ replacing $u$ and the prime indicating differentiation with regard to $y$. Further,

$$C_0^* = h^2\{(1-\tfrac{1}{2}hv_0)\,r_2 + (1+\tfrac{1}{2}hv_0)\,r_4 - 2r_0\}/12. \tag{32}$$

The addition of (31) to (16) eliminates the terms $\pm h^2 r_0$ and the only dependence of the equation which results from this elimination upon values of $r$ comes through the terms $C_0$ and $C_0^*$. Because these terms, given by (17d) and (32), respectively, are $O(h^4)$ we need employ only $h^2$-accurate expressions to evaluate the terms in $r$ which occur in them. This still preserves the $h^4$ accuracy of approximating (2).

Several methods can be used to form the compact approximation on the 9-point element of Fig. 1 but they are basically similar in principle. For example, the approximation (6) is $h^2$-accurate for any value of $\alpha$ and it may be used to eliminate $r_0$ from (32) in terms of $\phi_0$, $\phi_1$, and $\phi_3$. A like equation at the point $x = x_0$, $y = y_0 + h$ can be written down to express $r_2$ in terms of $\phi_2$, $\phi_5$, and $\phi_6$, with a comparable approximation at $x = x_0$, $y = y_0 - h$ expressing $r_4$ in terms of $\phi_4$, $\phi_7$, and $\phi_8$. Similarly, if we neglect $C_0^*$ in (31), the resulting $h^2$-accurate approximation

with two further ones at $x = x_0 \pm h$, $y = y_0$ may be used to eliminate $r_0$, $r_1$, and $r_3$ from (17d). Thus all terms in (17d) and (32) can in this way be expressed in terms of values of $\phi$ at nodes of the 9-point element.

The method of Dennis and Hudson [24] employed this procedure in principle; the central-difference approximations which correspond to $\alpha = 0$ in (6) and other similar formulae were used to evaluate $C_0$ and $C_0^*$. However, in [24, 25] the $h^4$-accurate terms in the final approximations were left as deferred corrections and all coefficients multiplying terms in $\phi_n$ were left in exponential form. In the present case one of the objections to using the corrections $C_0$ and $C_0^*$ in the respective discretized forms (17d) and (32) in the elimination process is that the expressions used to eliminate $r_1$, $r_2$, $r_3$, and $r_4$ will contain coefficients depending on values of $u$ and $v$ at points other than the central point $(x_0, y_0)$ of the 9-point molecule. This may not be desirable when these velocity components are obtained from the numerical solution of some other equation, e.g., from a stream function. We have therefore adopted a derivation of the compact approximation which depends upon non-discretized forms of the corrections $C_0$ and $C_0^*$. Some details are given in the Appendix for the sake of clarity of the approach used, but the final result of the elmination gives the $h^4$-accurate results (30), where the coefficients $d_n$ are given by

$$
\begin{aligned}
d_0 &= 40 + 2h^2(u_0^2 + v_0^2) - 4h^2[(\partial u/\partial x)_0 + (\partial v/\partial y)_0] \\
d_1 &= 8 - 4hu_0 + h^2[u_0^2 - 2(\partial u/\partial x)_0] + \tfrac{1}{2}h^3[u_0(\partial u/\partial x)_0 + v_0(\partial u/\partial y)_0 - (\nabla^2 u)_0] \\
d_2 &= 8 - 4hv_0 + h^2[v_0^2 - 2(\partial v/\partial y)_0] + \tfrac{1}{2}h^3[u_0(\partial v/\partial x)_0 + v_0(\partial v/\partial y)_0 - (\nabla^2 v)_0] \\
d_3 &= 8 + 4hu_0 + h^2[u_0^2 - 2(\partial u/\partial x)_0] - \tfrac{1}{2}h^3[u_0(\partial u/\partial x)_0 + v_0(\partial u/\partial y)_0 - (\nabla^2 u)_0) \\
d_4 &= 8 + 4hv_0 + h^2[v_0^2 - 2(\partial v/\partial y)_0] - \tfrac{1}{2}h^3[u_0(\partial v/\partial x)_0 + v_0(\partial v/\partial y)_0 - (\nabla^2 v)_0] \\
d_5 &= 2 - h(u_0 + v_0) + \tfrac{1}{2}h^2 u_0 v_0 - \tfrac{1}{2}h^2 H_0 \\
d_6 &= 2 + h(u_0 - v_0) - \tfrac{1}{2}h^2 u_0 v_0 + \tfrac{1}{2}h^2 H_0 \\
d_7 &= 2 + h(u_0 + v_0) + \tfrac{1}{2}h^2 u_0 v_0 - \tfrac{1}{2}h^2 H_0 \\
d_8 &= 2 - h(u_0 - v_0) - \tfrac{1}{2}h^2 u_0 v_0 + \tfrac{1}{2}h^2 H_0 .
\end{aligned}
\tag{33}
$$

In these expressions

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \tag{34}$$

and

$$H_0 = (\partial v/\partial x)_0 + (\partial u/\partial y)_0 . \tag{35}$$

The term $B_0 = 0$ in (30) when (33) are used to define the coefficients $d_n$ which give the $h^4$ approximation to (2). In the case of the equation

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 - u\partial\phi/\partial x - v\partial\phi/\partial y = R(x, y) \tag{36}$$

the coefficients in (30) are still given by (33) but with

$$B_0 = -h^2\{8R_0 + (1 - \tfrac{1}{2}hu_0)\,R_1 + (1 - \tfrac{1}{2}hv_0)\,R_2 + (1 + \tfrac{1}{2}hu_0)\,R_3 + (1 + \tfrac{1}{2}hv_0)\,R_4\}. \tag{37}$$

It may be noted that if $u = v = 0$ in (36), the approximation (30) gives the standard 9-point $h^4$ approximation to Poisson's equation. We could also point out that, although the last bracket in the expression for $d_0$ in (33) can be equated to zero in the case of an incompressible fluid, it does not seem desirable to do so. The reason for this is that this term balances the terms on the right-hand side of (30) which involve the quantities $(\partial u/\partial x)_0$ or $(\partial v/\partial y)_0$ in the coefficients $d_1$–$d_4$ in (33). The balance of all these terms on the left-hand side of (30) is $O(h^4)$ because of the second relationship of (18) (interpreted in turn in the $x$ and $y$ directions). The terms in $(\partial u/\partial x)_0$ and $(\partial v/\partial y)_0$ in the coefficients $d_1$–$d_4$ of (33) have to be calculated and they will involve $O(h^2)$ truncation errors. If, therefore, we omit the term in $(\partial u/\partial x)_0 + (\partial v/\partial y)_0$ in $d_0$, these truncation errors will be unbalanced and the approximation (30) will lose its $O(h^4)$ accuracy. It therefore seems to be appropriate to include the term $(\partial u/\partial x)_0 + (\partial v/\partial y)_0$ in (30) by actually calculating the derivatives with $h^2$-accurate central-difference formulae. In any case these derivatives are required in the calculation of $d_1$–$d_4$, so no additional computational work is involved.

Finally, there is an alternative way of expressing the coefficients (33) which is particularly useful if the Navier–Stokes equations are to be solved numerically using the formulation in terms of primitive variables. In such a case the function $\phi(x, y)$ would itself be a velocity component, for example, a typical equation would be

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - u\partial u/\partial x - v\partial u/\partial y = \partial p/\partial x, \tag{38}$$

where the term on the right-hand side is the pressure gradient in the $x$ direction and all variables are dimensionless. The $h^4$-accurate approximation to (38) would now be (30) with $u$ replacing $\phi$ but the coefficients $d_n$ given by (33) and there would be a similar equation to (38) defining the velocity component $v$ in terms of the pressure gradient $\partial p/\partial y$ leading to a similar approximation (30) with the same coefficients (33). The pressure gradient in (38) can be identified with the term $R(x, y)$ in (36), and hence $B_0$ in (30) is defined by (37).

The coefficients $d_1$–$d_4$ can then be rewritten as

$$\begin{aligned}
d_1 &= 8 - 4hu_0 + h^2u_0^2 - 2h^2(\partial u/\partial x)_0 - \tfrac{1}{2}h^3(\partial p/\partial x)_0 \\
d_2 &= 8 - 4hv_0 + h^2v_0^2 - 2h^2(\partial v/\partial y)_0 - \tfrac{1}{2}h^3(\partial p/\partial y)_0 \\
d_3 &= 8 + 4hu_0 + h^2u_0^2 - 2h^2(\partial u/\partial x)_0 + \tfrac{1}{2}h^3(\partial p/\partial x)_0 \\
d_4 &= 8 + 4hv_0 + h^2v_0^2 - 2h^2(\partial v/\partial y)_0 + \tfrac{1}{2}h^3(\partial p/\partial y)_0.
\end{aligned} \tag{39}$$

## 5. Computational Examples

Four numerical illustrations of the present $h^4$ methods are given in this section.

Example 1. The first illustration is the solution of the Blasius equation for boundary-layer flow past a semi-infinite flat plate. For steady flow this gives a 1-dimensional example which has been considered by Hirsh [12] during the course of his investigation of numerical results which can be obtained using $h^4$ implicit methods due to Kreiss.

The formulation of the problem and the variables used are given by Schlichting [28]. Here we use a variable $\eta$ which is half the corresponding variable used by Schlichting [28, p. 117]. The equation to be solved for the function $f(\eta)$ is then

$$f''' + ff'' = 0 \tag{40}$$

with the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 2. \tag{41}$$

If we put

$$f'(\eta) = 2 + \phi(\eta) \tag{42}$$

then (40) can be written as

$$\phi'' - u\phi' = 0 \tag{43}$$

with

$$u' = -(2 + \phi) \tag{44}$$

and boundary conditions

$$\phi(0) = -2, \qquad \phi(\infty) = 0, \qquad u(0) = 0. \tag{45}$$

The problem now gives a straightforward application of (21) with $r_0 = C_0 = 0$ at each grid point. The function $u(\eta)$ needed to define (19) is obtained by integrating (44) subject to the initial condition in (45). This calculation is performed step by step using the $h^4$-accurate approximation

$$u(h) = -2h - h\{9\phi(0) + 19\phi(h) - 5\phi(2h) + \phi(3h)\}/24 \tag{46}$$

for the first step and then continuing the calculation using Simpson's formula at uniformly spaced grid points from $\eta = 0$ to some large enough value $\eta = \eta_\infty$ at which the condition $\phi(\eta_\infty) = 0$ is a satisfactory approximation to the condition $\phi(\infty) = 0$ in (45).

A simple iterative successive over-relaxation procedure is used to solve the equa-

tions. With a suitable choice of relaxation parameter and value of $\eta_\infty$, one iterative sweep of (21) is followed by an application of (46) and continuation by Simpson's rule to determine an approximate $u(\eta)$ up to $\eta = \eta_\infty$. This process is repeated until

$$\sum |1 - \phi^{(k)}/\phi^{(k+1)}| < \varepsilon, \tag{47}$$

where $k$ is the iteration count and the summation is over all internal grid points. The procedure converges for all small enough $h$ because $u''(0) = -1.328$, approximately, and $u''(\eta)$ increases monotonically to zero as $\eta$ increases from $\eta = 0$ to $\eta = \infty$. Thus it is very easy to satisfy (22); for example, when $h = 0.4$ the left-hand side of (22) is never more than 0.085 which is many times less than the right-hand side within the necessary computational domain.

The calculations were carried out for $h = 0.1$, 0.2, 0.4 and 0.8 taking $\eta_\infty = 6.4$ which is considerably greater than necessary; i.e., it corresponds to the value $\eta = 12.8$ of the variable $\eta$ used by Schlichting (cf. Table 7.1 of Ref. [28]). The value $\varepsilon = 10^{-8}$ was used in (47). The minimum number of iterations occurred with a relaxation factor of about 1.7 and for this value the number of iterations required starting from the initial assumption $\phi(\eta) \equiv 0$ $(\eta > 0)$, $u(\eta) = -2\eta$ was in the neighborhood of 50 for $h = 0.2$, 0.4, 0.8 and about 180 for $h = 0.1$.

Some typical results are shown in Table I which gives values of $f(\eta)$ and $f'(\eta)$ at several locations of $\eta$ for the four grids together with the corresponding values of $f''(0)$. This is obtained by integrating (43) from $\eta = 0$ to $\eta = \infty$, which gives

$$f''(0) = \phi'(0) = -\int_0^\infty \phi(2 + \phi)\, d\eta. \tag{48}$$

The integral is then evaluated by Simpson's formula. Because of a difference in definition of variables, the grid used by Hirsh [12] appears to be $H = h\sqrt{2}$ so it is not possible to compare directly with results in Table I. For the sake of comparison we have computed results using four grid sizes equal to four used by Hirsh. Values of $f''(0)/(2\sqrt{2})$ from the present solutions are compared with those

TABLE I

Calculated Approximations to the Blasius Function $f(\eta)$ and Its Derivatives Using the $h^4$-Accurate Approximation (21)

| $h$ | $f''(0)$ | $f(0.8)$ | $f'(0.8)$ | $f(1.6)$ | $f'(1.6)$ | $f(2.4)$ | $f'(2.4)$ |
|---|---|---|---|---|---|---|---|
| 0.8 | 1.35939 | 0.4306443 | 1.0377541 | 1.5747490 | 1.7542925 | 3.1048803 | 1.9734612 |
| 0.4 | 1.32911 | 0.4200633 | 1.0333462 | 1.5689592 | 1.7520514 | 3.0850878 | 1.9754066 |
| 0.2 | 1.32828 | 0.4203061 | 1.0335035 | 1.5690861 | 1.7521553 | 3.0853062 | 1.9755678 |
| 0.1 | 1.32823 | 0.4203199 | 1.0335131 | 1.5690937 | 1.7521620 | 3.0853195 | 1.9755783 |
| 0.05 | 1.32823 | 0.4203207 | 1.0335131 | 1.5690947 | 1.7521620 | 3.0853205 | 1.9755783 |

TABLE II

Comparisons of the Skin Friction from the Blasius Equation Using Grids $H = h\sqrt{2}$

| | $0.469600 - f''(0)/2^{3/2}$ | |
|---|---|---|
| $H$ | Present | Hirsh [10] |
| 1.0 | −0.004064 | 0.038766 |
| 0.5 | −0.000110 | 0.003887 |
| 0.4 | −0.000043 | 0.001170 |
| 0.2 | −0.000002 | 0.000018 |

calculated by Hirsh in Table II. Hirsh gives the exact value $E = f''(0)/(2\sqrt{2})$ as $E = 0.469600$ which we have confirmed by means of the solution at $h = 0.05$. Calculated results at various grid sizes are compared with $E$ in Table II for the solutions of Hirsh and the present solutions.

EXAMPLE 2. As a simple example of a time-dependent problem in one space dimension we consider the equation

$$\partial\phi/\partial t = \partial^2\phi/\partial x^2 + 2x\,\partial\phi/\partial x \tag{49}$$

with the initial condition and boundary conditions

$$\begin{aligned} \phi &= 0 \qquad \text{when} \quad t = 0,\ x > 0 \\ \phi(0, t) &= 1, \qquad \phi(\infty, t) = 0, \qquad t > 0. \end{aligned} \tag{50}$$

There is a step discontinuity at $x = 0$ when $t = 0$, which means that considerable effort is necessary to obtain an accurate solution near $x = 0$ for small $t > 0$. As $t \to \infty$, $\phi(x, t) \to \phi(x, \infty)$ given by

$$\phi(x, \infty) = 1 - \operatorname{erf} x, \tag{51}$$

which gives an exact solution to check the accuracy of the steady state solution.

In fact there is an exact solution in series form to the time-dependent equation (49) subject to (50) which is given by

$$\phi(x, t) = 1 - \operatorname{erf} x + \pi^{-1/2} e^{-x^2} \sum_{p=1}^{\infty} (-1)^p e^{-4pt} H_{2p-1}(x)/(2^{2p-1}\, p!). \tag{52}$$

Here the functions of $x$ in the summation are the Hermite polynomials. Many terms of the series in (52) are necessary to obtain an accurate solution near $t = 0$.

Our present purpose is to check that, for a given method of truncation in time, the approximation to the solution of (49) obtained using the $h^4$-accurate

TABLE III

Computed Solutions of (49)

| | | $x=0.2$ | | | $x=0.8$ | | | $x=1.6$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | $h$ | $\phi_A$ | $\phi_B$ | $E$ | $\phi_A$ | $\phi_B$ | $E$ | $\phi_A$ | $\phi_B$ | $E$ |
| 0.05 | 0.2 | 0.4995 | 0.505615 | 0.506286 | 0.0119 | 0.007910 | 0.007849 | 0.0000 | 0.000000 | 0.000000 |
| | 0.1 | 0.5045 | 0.506244 | 0.506286 | 0.0090 | 0.007853 | 0.007849 | 0.0000 | 0.000000 | 0.000000 |
| 0.2 | 0.2 | 0.7024 | 0.703757 | 0.703230 | 0.1258 | 0.128403 | 0.127554 | 0.0025 | 0.002380 | 0.002306 |
| | 0.1 | 0.7005 | 0.703263 | 0.703230 | 0.1268 | 0.127607 | 0.127554 | 0.0023 | 0.002311 | 0.002306 |
| 0.5 | 0.2 | 0.7599 | 0.761090 | 0.761020 | 0.2222 | 0.223942 | 0.223768 | 0.0150 | 0.015030 | 0.014968 |
| | 0.1 | 0.7607 | 0.761024 | 0.761020 | 0.2232 | 0.223779 | 0.223768 | 0.0150 | 0.014972 | 0.014968 |
| 1.0 | 0.2 | 0.7749 | 0.775294 | 0.775276 | 0.2532 | 0.253540 | 0.253489 | 0.0228 | 0.022413 | 0.022382 |
| | 0.1 | 0.7752 | 0.775277 | 0.775276 | 0.2534 | 0.253492 | 0.253489 | 0.0225 | 0.022384 | 0.022382 |
| 4.0 | 0.2 | 0.7771 | 0.777296 | 0.777297 | 0.2581 | 0.257910 | 0.257898 | 0.0242 | 0.023671 | 0.023652 |
| | 0.1 | 0.7773 | 0.777297 | 0.777297 | 0.2579 | 0.257899 | 0.257898 | 0.0238 | 0.023653 | 0.023652 |

*Note.* $\phi_A$ is the $h^2$-accurate solution obtained by applying the Crank–Nicolson method to (6) with $\alpha=\frac{1}{8}$. $\phi_B$ is the $h^4$-accurate solution obtained using (21). $E$ is the exact solution.

approximation in space is superior to that obtained using $h^2$-accurate spatial approximations. Naturally, the accuracy of the numerical solution will depend also upon the accuracy of the method of approximation in time. In the present paper the Crank–Nicolson method outlined in Section 3 was used in the following way. Numerical solutions were obtained for two separate spatial grid sizes $h=0.1$, 0.2 with $\phi(x_\infty, t)=0$, where $x_\infty=4.8$ which is more than large enough to approximate adequately the condition on $\phi$ for large $x$. For each spatial grid the time step was reduced sufficiently (by repeatedly halving it) until the solution has converged to a limit at all grid points. In this way the numerical solutions were made independent of the truncation in time.

Some results of these calculations are shown in Table III, where comparisons are made at some selected values of the time. Both the $h^2$-accurate approximation (6) with the value $\alpha=\frac{1}{8}$ proposed by Dennis and Hudson [4] and the $h^4$-accurate approximation of the present paper were used; these form the basis of comparison in Table III and they show clearly the superiority of the $h^4$-accurate method. It is also of passing interest to note that the $h^2$-accurate central-difference approximation which corresponds to $\alpha=0$ in (6) was also used and found to give slightly inferior results to the $h^2$-accurate results in Table III.

EXAMPLE 3. As an example involving two space variables in a steady state problem we consider again an example used by Dennis and Hudson [24] to illustrate the $h^4$-accurate method in which an equation of type (36) was

approximated using present methods but the final approximation was left with exponential coefficients which were not expanded further as at present. This example considered the simultaneous equations

$$\begin{aligned} \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 - u\partial u/\partial x - v\partial u/\partial y &= (2\sin y + \sin x)\cos x \\ \partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 - u\partial v/\partial x - v\partial v/\partial y &= (\sin y - 2\sin x)\cos y \end{aligned} \tag{53}$$

which have a simple exact solution

$$u(x, y) = -\cos x \sin y, \qquad v(x, y) = \sin x \cos y. \tag{54}$$

The solution of (53) was sought within the square region $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$ subject to boundary conditions for $u$ and $v$ calculated from (54) on the square boundaries.

This problem is a modification of one considered by Roscoe [29] in a discussion of numerical methods very similar to those of Allen and Southwell [30]. The terms on the right-hand side of (53) do not themselves constitute possible pressure gradients for (53) to be truly representative of the 2-dimensional Navier–Stokes equations, but (54) satisfy the equation of continuity. Thus the problem of solving numerically (53) gives a satisfactory enough simple application of the present methods. Dennis and Hudson [24, pp. 49–50] considered approximate solutions by several methods and found that the $h^4$ difference equations with exponential coefficients gave very accurate results. It was in fact possible to obtain a solution correct to between four and five decimal accuracy using a grid size of $h = \pi/10$.

In the present paper Eqs. (53) have been solved using the methods of Section 4 with the same grid $h = \pi/10$ and employing a simple iterative successive over-relaxation method in which one sweep of the domain corresponding to the first of (53) is followed by a similar sweep of the second. The procedure was started from the assumption that $u = v = 0$ everywhere except on the boundaries and converged rapidly to a solution to five decimal precision. In Table IV we give details of four computed solutions for $u(0.7\pi, y)$ for values of $y/\pi$ from 0 to 0.5. The solution $u_A$

TABLE IV

Comparison of the $h^4$-Approximations $u_B$ Obtained by Present Methods and $u_C$ in [24] with Exact Solution $u_A$ to (53) with (54) at $x = 0.7\pi$

| $y/\pi$ | $u_A$ | $u_B$ | $u_C$ | $u_D$ | $u_E$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.18164 | 0.18163 | 0.18164 | 0.1817 | 0.1829 |
| 0.2 | 0.34549 | 0.34548 | 0.34550 | 0.3452 | 0.3478 |
| 0.3 | 0.47553 | 0.47553 | 0.47557 | 0.4749 | 0.4785 |
| 0.4 | 0.55902 | 0.55903 | 0.55909 | 0.5580 | 0.5624 |
| 0.5 | 0.58779 | 0.58781 | 0.58787 | 0.5866 | 0.5913 |

*Note.* The approximations $u_D$ and $u_E$ are $h^2$ accurate.

is the exact solution and $u_B$ is the $h^4$-accurate solution obtained by means of the approximations given in Section 4. The solution $u_C$ is the solution computed by Dennis and Hudson [24] by means of $h^4$-accurate approximations with exponential coefficients. Finally, $u_D$ and $u_E$ are $h^2$-accurate solutions obtained using the 2-dimensional analogue of (6) with the respective values $\alpha = 1/12$ for $u_D$ and $\alpha = 0$ (central differences) for $u_E$. One could observe that the present results are an improvement on those given by Dennis and Hudson [24], i.e., $u_B$ is generally an improvement on $u_C$ and both are substantially better than the results obtained from the $h^2$-accurate approximations.

In this example the exact solution is known and we can demonstrate numerically that our finite difference method is $h^4$-accurate. With this objective, we repeated the calculations above with (i) $h = \pi/10$ and (ii) $h = \pi/20$ using double precision arithmetic and assuming that the iteration had converged when

$$\sum \left| 1 - \frac{u^{(k)}}{u^{(k+1)}} \right|, \qquad \sum \left| 1 - \frac{v^{(k)}}{v^{(k+1)}} \right| \leqslant 10^{-8},$$

where $k$ is an iteration count and the summations include all internal points of the solution domain. Dividing the error in the coarse mesh solution at a given point by the corresponding error in the fine mesh solution gave a value of approximately 16—as one would expect from an $h^4$-accurate method when the steplength is halved. Typical results are given in Table V.

The exact solution to the time dependent problem of Example 2 is also known and it is easily seen from the results given in Table III that the error in $\phi_B(x, t)$ is reduced by a factor of approximately 16 when the steplength in the $x$-direction is reduced from 0.2 to 0.1.

For the problem discussed in Example 1, we note that the values of $f'(\eta)$ obtained by the $h^4$-accurate method with $h = 0.1$ and $h = 0.05$ agree exactly to the eight significant figures quoted in Table I. If we assume that these values represent the exact solution, then again we find that the errors are reduced by a factor of

TABLE V

Numerical Demonstration of the Order of the Finite-Difference Approximation when Applied to Example 3

| $y/\pi$ | $u_A$ | $u_B$ | $u_B^*$ | $\frac{\text{Error in } u_B}{\text{Error in } u_B^*}$ |
|---|---|---|---|---|
| 0.1 | 0.18163563 | 0.18165482 | 0.18163680 | 16.4 |
| 0.2 | 0.34549150 | 0.34551853 | 0.34549315 | 16.4 |
| 0.3 | 0.47552826 | 0.47555596 | 0.47552993 | 16.6 |
| 0.4 | 0.55901699 | 0.55904263 | 0.55901853 | 16.7 |
| 0.5 | 0.58778525 | 0.58780972 | 0.58778671 | 16.8 |

Note. $u_A$ is the exact solution and $u_B$, $u_B^*$ are the numerical solutions with $h = \pi/10$, $\pi/20$, respectively.

approximately 16 when the steplength is reduced from 0.4 to 0.2. The factor is less consistent when the steplength is reduced from 0.8 to 0.4, but even in this case, the results indicate that the order of the method is in the neighbourhood of four.

EXAMPLE 4. As a final example we consider the problem of 2-dimensional free heat convection in a square cavity defined by $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. All variables are assumed to be dimensionless. In terms of these variables the governing equations can be written

$$\nabla^2 \psi = -\zeta, \tag{55a}$$

$$\nabla^2 \zeta = \mathrm{Pr}^{-1}\left(u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right) - \mathrm{Ra}\frac{\partial T}{\partial x}, \tag{55b}$$

$$\nabla^2 T = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}, \tag{55c}$$

where $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ are the 2-dimensional velocity components in the $x$ and $y$ directions, Ra is the Rayleigh number, Pr is the Prandtl number, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

If $C$ denotes the unit square and $\mathbf{n}$ is the outward normal to it at any point, Eqs. (55) are to be solved within $C$ subject to the conditions

$$\psi = \partial\psi/\partial n = 0 \qquad \text{on} \quad C; \tag{56a}$$

$$T = 1 \text{ when } x = 0, \qquad T = 0 \text{ when } x = 1; \tag{56b}$$

$$\partial T/\partial y = 0 \qquad \text{when} \quad y = 0,\ 1. \tag{56c}$$

The three equations (55) are each of the type (36) and can thus be approximated by the $h^4$ formulae discussed in Section 4. The term involving $\partial T/\partial x$ in (55b) is a forcing term coming from the solution of (55c) and must be approximated in (55b) correct to $h^4$ accuracy. However, if we rewrite Eq. (37) in the form

$$B_0 = -h^2\left\{12R_0 + [R_1 + R_2 + R_3 + R_4 - 4R_0] - \frac{1}{2}h[u_0(R_1 - R_3) + v_0(R_2 - R_4)]\right\}, \tag{57}$$

it is necessary to use a 5-point formula only for estimating $R_0 = -\mathrm{Ra}(\partial T/\partial x)_0$, the standard 3-point central-difference approximation being sufficiently accurate for $R_1$, $R_2$, $R_3$, and $R_4$.

An expression for the boundary vorticity, given by Woods [31], may be written in the form

$$\zeta_B = -\frac{3}{h^2}\psi_{I_1} - \frac{1}{2}\zeta_{I_1} + O(h^2), \tag{58}$$

where $B$ denotes a point on the boundary and $I_1$ denotes the first internal point on the normal through $B$. For the present investigation it is necessary to compute the $O(h^2)$ and $O(h^3)$ terms and this leads to

$$\zeta_B = -\frac{15}{23h^2}(8\psi_{I_1} - \psi_{I_2}) - \frac{1}{23}(16\zeta_{I_1} - 11\zeta_{I_2} + 2\zeta_{I_3}) + O(h^4), \tag{59}$$

where $I_2$ and $I_3$ are the next two internal points on the normal through $B$.

We obtained solutions for $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^3, 10^4, 10^5$ with $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \frac{1}{40}$ using both the $h^2$-accurate 2-dimensional form of (6) with $\alpha = \frac{1}{8}$ and the $h^4$-accurate method of Section 4. In each case the program performed one iteration of the vorticity field followed by $n$ iterations of the streamfunction and temperature fields and repeated this until

$$\sum \left| 1 - \frac{\zeta^{(k)}}{\zeta^{(k+1)}} \right|, \qquad \sum \left| 1 - \frac{\psi^{(k)}}{\psi^{(k+1)}} \right|, \qquad \sum \left| 1 - \frac{T^{(k)}}{T^{(k+1)}} \right| \leqslant 10^{-5}.$$

In general we set $n = 1$; however, for the $O(h^4)$ calculations with $\mathrm{Ra} = 10^5$ we increased $n$ to around 25 and introduced a relaxation factor of 0.3 to ensure convergence of the iterative procedure. If solutions at higher Rayleigh numbers were required, it would probably be necessary to rewrite the $O(h^4)$ method as an $O(h^3)$ method with difference correction as described in Section 2.

The results in Tables VI–VIII were obtained using the 2-dimensional form of (6)

TABLE VI

Properties of the Solution of the Heat Convection Problem Obtained Using the 2-Dimensional $h^2$-Accurate Analogue of (6) for the Case $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^3$

| $h$ | $\lvert\psi_{\mathrm{mid}}\rvert$ | $u_{\max}$ $y\,(x = 0.5)$ | $v_{\max}$ $x\,(y = 0.5)$ | $Nu_0$ | $Nu_{\max}$ $y\,(x = 0)$ | $Nu_{\min}$ $y\,(x = 0)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 1.171 | 3.398<br>0.802 | 3.421<br>0.193 | 1.106 | 1.453<br>0.121 | 0.728<br>1 |
| $\frac{1}{20}$ | 1.171 | 3.580<br>0.811 | 3.621<br>0.181 | 1.112 | 1.485<br>0.097 | 0.703<br>1 |
| $\frac{1}{30}$ | 1.173 | 3.618<br>0.812 | 3.662<br>0.180 | 1.115 | 1.495<br>0.092 | 0.697<br>1 |
| $\frac{1}{40}$ | 1.174 | 3.631<br>0.813 | 3.677<br>0.179 | 1.116 | 1.500<br>0.090 | 0.694<br>1 |
| Bench-mark solution | 1.174 | 3.649<br>0.813 | 3.697<br>0.178 | 1.117 | 1.505<br>0.0092 | 0.692<br>1 |

TABLE VII

Properties of the Solution of the Heat Convection Problem Obtained Using the 2-Dimensional $h^2$-Accurate Analogue of (6) for the Case $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^4$

| $h$ | $\lvert\psi_{\mathrm{mid}}\rvert$ | $u_{\max}$ $y\,(x=0.5)$ | $v_{\max}$ $x\,(y=0.5)$ | $Nu_0$ | $Nu_{\max}$ $y\,(x=0)$ | $Nu_{\min}$ $y\,(x=0)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 5.282 | 15.435<br>0.809 | 17.258<br>0.041 | 2.294 | 3.797<br>0.170 | 0.620<br>1 |
| $\frac{1}{20}$ | 5.117 | 15.972<br>0.820 | 19.014<br>0.125 | 2.238 | 3.553<br>0.151 | 0.599<br>1 |
| $\frac{1}{30}$ | 5.091 | 16.083<br>0.822 | 19.344<br>0.122 | 2.234 | 3.521<br>0.148 | 0.592<br>1 |
| $\frac{1}{40}$ | 5.083 | 16.125<br>0.822 | 19.465<br>0.120 | 2.236 | 3.518<br>0.147 | 0.589<br>1 |
| Bench-mark solution | 5.071 | 16.178<br>0.823 | 19.617<br>0.119 | 2.238 | 3.528<br>0.143 | 0.586<br>1 |

TABLE VIII

Properties of the Solution of the Heat Convection Problem Obtained Using the 2-Dimensional $h^2$-Accurate Analogue of (6) for the Case $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^5$

| $h$ | $\lvert\psi_{\mathrm{mid}}\rvert$ | $u_{\max}$ $y\,(x=0.5)$ | $v_{\max}$ $x\,(y=0.5)$ | $Nu_0$ | $Nu_{\max}$ $y\,(x=0)$ | $Nu_{\min}$ $y\,(x=0)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 11.707 | 38.29<br>0.849 | 54.58<br>0.091 | 4.493 | 7.813<br>0.186 | 0.708<br>1 |
| $\frac{1}{20}$ | 9.638 | 35.44<br>0.854 | 62.76<br>0.076 | 4.827 | 9.040<br>0.105 | 0.722<br>1 |
| $\frac{1}{30}$ | 9.311 | 34.87<br>0.855 | 66.26<br>0.071 | 4.619 | 8.194<br>0.090 | 0.729<br>1 |
| $\frac{1}{40}$ | 9.215 | 34.76<br>0.855 | 67.16<br>0.068 | 4.553 | 7.902<br>0.087 | 0.730<br>1 |
| Bench-mark solution | 9.111 | 34.73<br>0.855 | 68.59<br>0.066 | 4.509 | 7.717<br>0.081 | 0.729<br>1 |

with $\alpha = \frac{1}{8}$; in this case the boundary vorticity was evaluated from (58) and the standard 3-point approximation was used to represent $\mathrm{Ra}(\partial T/\partial x)$ and condition (56c). The results in Tables IX–XI were obtained using the scheme defined by Eqs. (30), (33), and (57); in this case the boundary vorticity was evaluated from (59) and a 5-point approximation was used to represent condition (56c).

In each table we give

(i) the magnitude of the streamfunction at the mid-point of the cavity

(ii) the maximum value of $u$ on the vertical mid-plane, together with its location

(iii) the maximum value of $v$ on the horizontal mid-plane, together with its location

(iv) the average Nusselt number $Nu_0$, on the vertical boundary at $x = 0$

(v) the maximum and minimum values of the local Nusselt number on $x = 0$, together with their locations.

Note that the maximum values referred to above (and their locations) were evaluated by using a fourth-order interpolating polynomial.

Recent bench-mark solutions of this problem have been given by de Vahl Davis [32] and various comparison solutions have been discussed by de Vahl Davis and Jones [33]. The bench-mark solutions were obtained by using mesh refinement and extrapolation in conjunction with a second-order method and are claimed to be very accurate. These solutions are therefore included in Tables VI–XI for

TABLE IX

Properties of the Solution of the Heat Convection Problem Obtained Using the $h^4$-Accurate Methods of Section 4 for the Case $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^3$

| $h$ | $\lvert\psi_{\mathrm{mid}}\rvert$ | $u_{\max}$<br>$y\,(x=0.5)$ | $v_{\max}$<br>$x\,(y=0.5)$ | $Nu_0$ | $Nu_{\max}$<br>$y\,(x=0)$ | $Nu_{\min}$<br>$y\,(x=0)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 1.1784 | 3.6702<br>0.8120 | 3.7179<br>0.1803 | 1.1167 | 1.4997<br>0.0690 | 0.06946<br>1 |
| $\frac{1}{20}$ | 1.1752 | 3.6516<br>0.8133 | 3.6998<br>0.1783 | 1.1167 | 1.5039<br>0.0815 | 0.6914<br>1 |
| $\frac{1}{30}$ | 1.1748 | 3.6500<br>0.8133 | 3.6980<br>0.1783 | 1.1173 | 1.5053<br>0.0858 | 0.6913<br>1 |
| $\frac{1}{40}$ | 1.1747 | 3.6497<br>0.8132 | 3.6977<br>0.1783 | 1.1176 | 1.5058<br>0.0871 | 0.6913<br>1 |
| Bench-mark solution | 1.174 | 3.649<br>0.813 | 3.697<br>0.178 | 1.117 | 1.505<br>0.092 | 0.692<br>1 |

TABLE X

Properties of the Solution of the Heat Convection Problem Obtained Using the $h^4$-Accurate Methods of Section 4 for the Case $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^4$

| $h$ | $\lvert\psi_{\mathrm{mid}}\rvert$ | $u_{\max}$ $y\,(x=0.5)$ | $v_{\max}$ $x\,(y=0.5)$ | $Nu_0$ | $Nu_{\max}$ $y\,(x=0)$ | $Nu_{\min}$ $y\,(x=0)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 5.0996 | 16.2845<br>0.8212 | 19.3361<br>0.1273 | 2.3305 | 3.7527<br>0.1737 | 0.5926<br>1 |
| $\frac{1}{20}$ | 5.0735 | 16.1842<br>0.8232 | 19.6584<br>0.1191 | 2.2324 | 3.5093<br>0.1411 | 0.5858<br>1 |
| $\frac{1}{30}$ | 5.0734 | 16.1829<br>0.8232 | 19.6315<br>0.1189 | 2.2356 | 3.5110<br>0.1433 | 0.5853<br>1 |
| $\frac{1}{40}$ | 5.0735 | 16.1829<br>0.8232 | 19.6293<br>0.1189 | 2.2396 | 3.5193<br>0.1440 | 0.5851<br>1 |
| Bench-mark solution | 5.071 | 16.178<br>0.823 | 19.617<br>0.119 | 2.238 | 3.528<br>0.143 | 0.586<br>1 |

TABLE XI

Properties of the Solution of the Heat Convection Problem Obtained Using the $h^4$-Accurate Methods of Section 4 for the case $\mathrm{Pr} = 0.71$, $\mathrm{Ra} = 10^5$

| $h$ | $\lvert\psi_{\mathrm{mid}}\rvert$ | $u_{\max}$ $y(x=0.5)$ | $v_{\max}$ $x\,(y=0.5)$ | $Nu_0$ | $Nu_{\max}$ $y\,(x=0)$ | $Nu_{\min}$ $y\,(x=0)$ |
|---|---|---|---|---|---|---|
| $\frac{1}{10}$ | 9.4257 | 35.834<br>0.8552 | 52.781<br>0.0845 | 4.8230 | 8.6502<br>0.1778 | 0.7406<br>1 |
| $\frac{1}{20}$ | 9.1037 | 34.591<br>0.8536 | 68.083<br>0..0682 | 4.6880 | 8.3721<br>0.0988 | 0.7299<br>1 |
| $\frac{1}{30}$ | 9.1085 | 34.683<br>0.8544 | 68.700<br>0.0663 | 4.5220 | 7.8276<br>0.0800 | 0.7276<br>1 |
| $\frac{1}{40}$ | 9.1126 | 34.716<br>0.8545 | 68.637<br>0.0660 | 4.4959 | 7.6830<br>0.0800 | 0.7279<br>1 |
| Bench-mark solution | 9.111 | 34.73<br>0.855 | 68.59<br>0.066 | 4.509 | 7.717<br>0.081 | 0.729<br>1 |

TABLE XII

Percentage Differences between Various Properties of the Bench-Mark Solutions and Those of the Present $h^4$-Accurate Solutions

| Ra | $h$ | $\lvert\psi_{\text{mid}}\rvert$ | $u_{\max}$ | $v_{\max}$ | $Nu_0$ |
|---|---|---|---|---|---|
| $10^3$ | 1/10 | 0.37 | 0.58 | 0.57 | 0.03 |
| | 1/20 | 0.10 | 0.07 | 0.08 | 0.03 |
| | 1/30 | 0.07 | 0.03 | 0.03 | 0.03 |
| | 1/40 | 0.06 | 0.02 | 0.02 | 0.05 |
| $10^4$ | 1/10 | 0.56 | 0.66 | 1.43 | 4.13 |
| | 1/20 | 0.05 | 0.04 | 0.21 | 0.25 |
| | 1/30 | 0.05 | 0.03 | 0.07 | 0.11 |
| | 1/40 | 0.05 | 0.03 | 0.06 | 0.07 |
| $10^5$ | 1/10 | 3.45 | 3.18 | 23.05 | 6.96 |
| | 1/20 | 0.08 | 0.40 | 0.74 | 3.97 |
| | 1/30 | 0.03 | 0.14 | 0.16 | 0.29 |
| | 1/40 | 0.02 | 0.04 | 0.07 | 0.29 |

comparison purposes: the improved accuracy of the $O(h^4)$ results in tables IX–XI over the $O(h^2)$ results in Tables VI–VIII is immediately obvious.

To obtain a clearer assessment of our $O(h^4)$ results we give, in Table XII, the percentage differences between various properties of our solutions and those of the bench-mark solutions. De Vahl Davies [32] has estimated that the percentage errors in the latter for $\text{Ra} = 10^3$, $10^4$, and $10^5$ are no more than 0.1, 0.2, and 0.3, respectively. From Table XII we see that our results based on $h = \frac{1}{30}$ and $h = \frac{1}{40}$ are well within these tolerances for all Rayleigh numbers considered. Even with $h = \frac{1}{20}$, most of the properties for $\text{Ra} = 10^3$ and $\text{Ra} = 10^4$ are within the specified tolerances. It is relevant to note that the extrapolated bench-mark solutions were based on two $O(h^2)$ solutions obtained with (i) $h = \frac{1}{20}, \frac{1}{40}$ when $\text{Ra} = 10^3$, $10^4$ and (ii) $h = \frac{1}{40}, \frac{1}{80}$ when $\text{Ra} = 10^5$. Thus the $h^4$-accurate method gives results of comparable accuracy on a single comparatively coarse mesh.

In the first three examples considered, we noted that the error decreased by a factor of approximately 16 when the steplength was halved—as we would expect from an $O(h^4)$ method. In cases where the exact solution is unknown, such as the present example, it is possible to estimate the order of the method numerically provided that three solutions based on sufficiently small steplengths are available. However, when de Vahl Davies [32] applied the technique to his $O(h^2)$ solutions, he found that his estimate of the order varied throughout the cavity from just under one to just over three. Using our $O(h^4)$ results, we too found that the order varied throughout the cavity and conclude, with de Vahl Davies, that the steplengths used are too large for the procedure to be valid in this example.

Another difficulty, also described by de Vahl Davies, concerns the estimation of the local Nusselt numbers on the boundary at $x = 0$. He found that the use of

differentiation formulae of orders one to four produced significantly different values, particularly when the steplength and Rayleigh number were large, and concluded that the second-order formulae produced the best overall results. We carried out similar experiments on our $O(h^4)$ results but, in our case, the formulae of orders three and four were in closest agreement, with the third-order formula being marginally superior overall.

## Appendix

One method of obtaining (16) with the associated coefficients (17) is to use the result

$$\chi_1 + \chi_3 - 2\chi_0 = h^2\chi_0'' + \tfrac{1}{12}h^4\chi^{\mathrm{IV}} + O(h^6) \tag{A1}$$

and to substitute for the derivatives on the right-hand side using (12). With neglect of the $O(h^6)$ term in (A1) and use of (10), this gives the approximation

$$g_1\phi_1 + g_3\phi_3 - (2 - h^2 f_0)\,\phi_0 - h^2 r_0 + \frac{1}{12}h^4\left[\frac{d^2}{dx^2}(f\phi g - rg)\right]_0 = 0. \tag{A2}$$

The approximation (15) is obtained by expressing the second derivative in (A2) in central differences, but we can also evaluate this term by formal differentiation. It may be noted from (11) that

$$g' = -\tfrac{1}{2}ug; \qquad g(x_0) = 1 \tag{A3}$$

and we find that

$$\left[\frac{d^2}{dx^2}\left((f\phi g - rg)\right)\right]_0 = 2f_0\phi_0'' + 2(f_0' - u_0 f_0)\,\phi_0' + (f_0'' - u_0 f_0' - f_0^2)\,\phi_0 - (r_0'' - u_0 r_0'). \tag{A4}$$

The first two terms of (A2) can be dealt with by expanding $g_1\phi_1 + g_3\phi_3$ as a Taylor series

$$g_1\phi_1 + g_3\phi_3 = (\phi_1 + \phi_3)\left(1 + \frac{h^2}{2!}g_0'' + \frac{h^4}{4!}g_0^{\mathrm{IV}}\right) + (\phi_1 - \phi_3)\left(hg_0' + \frac{h^3}{3!}g_0'''\right) + O(h^6). \tag{A5}$$

The derivatives of $g$ in (A5) can easily be evaluated from (A3) to give, with neglect of the $O(h^6)$ term,

$$g_1\phi_1 + g_3\phi_3 = (\phi_1 + \phi_3)\left\{1 - \frac{1}{2}h^2 f_0 - \frac{h^4}{4!}(f_0'' - u_0 f_0' - f_0^2)\right\}$$
$$-(\phi_1 - \phi_3)\left\{\frac{1}{2}u_0 h + \frac{h^3}{3!}\left(f_0' - \frac{1}{2}u_0 f_0\right)\right\}. \tag{A6}$$

We now substitute (A4) and (A6) into the appropriate terms in (A2). The combination of the term multiplying $\phi_0$ from (A4) with the similar one multiplying $(\phi_1 + \phi_3)$ from (A6) can be omitted since $\phi_1 + \phi_3 - 2\phi_0 = O(h^2)$ and the resulting term is thus $O(h^6)$. Further, all remaining derivatives of $\phi$ and $r$ in (A2) can be expressed in $h^2$-accurate central differences without introduction of errors of order less than $O(h^6)$. Thus we obtain the formulae (17); the formula (17d) corresponds to the terms involving the derivatives of $r$ in (A4).

The form (21) of the approximation with the associated coefficients (19) is not quite the same as (16) with (17) but is still of $h^4$ accuracy. In fact, the sum of the three terms in $\phi_0, \phi_1$, and $\phi_3$ in (21) differs from the sum of the corresponding terms in (16) by terms of amount

$$h^4 u_0'^2(\phi_1 + \phi_3 - 2\phi_0)/144.$$

These are $O(h^6)$ accuracy, giving the required $h^4$-accurate approximation.

Finally, it has been pointed out in Section 4 that the most appropriate way of expressing the corrections $C_0$ and $C_0^*$ in terms of values of $\phi$ at the nodal points of the 9-point molecule seems to be to express them in non-discretized form. Thus we can write, approximately,

$$C_0 = -h^4[\partial^2 r/\partial x^2 - u\partial r/\partial x]_0/12 \tag{A7}$$

and if we use the definition

$$r(x, y) = -\partial^2\phi/\partial y^2 + v\partial\phi/\partial y \tag{A8}$$

from (3) and substitute in (A7), an expression for $C_0$ in terms of fourth and lower order derivatives of $\phi$ is found. Likewise, we can write

$$C_0^* = h^4[\partial^2 r/\partial y^2 - v\partial r/\partial y]_0/12 \tag{A9}$$

and substitute

$$r(x, y) = \partial^2\phi/\partial x^2 - u\partial\phi/\partial x \tag{A10}$$

in it to find a similar expression for $C_0^*$.

All the derivatives of $\phi$ in these two expressions are now discretized using $h^2$-accurate central-difference expressions. Some typical approximations needed are

$$h^4(\partial^4\phi/\partial x^2\partial y^2)_0 = 4\phi_0 - 2(\phi_1+\phi_2+\phi_3+\phi_4) + (\phi_5+\phi_6+\phi_7+\phi_8), \quad \text{(A11)}$$

$$2h^3(\partial^3\phi/\partial x^2\partial y)_0 = -2(\phi_2-\phi_4) + \phi_5 + \phi_6 - \phi_7 - \phi_8, \quad \text{(A12)}$$

$$2h^3(\partial^3\phi/\partial x\,\partial y^2)_0 = -2(\phi_1-\phi_3) + \phi_5 - \phi_6 - \phi_7 + \phi_8, \quad \text{(A13)}$$

$$4h^2(\partial^2\phi/\partial x\,\partial y)_0 = \phi_5 - \phi_6 + \phi_7 - \phi_8, \quad \text{(A14)}$$

together with the results similar to (18) in terms of derivatives of $\phi$ in the $x$ direction, with corresponding results in terms of $\phi_2$, $\phi_2$, and $\phi_4$ for derivatives of $\phi$ in the $y$ direction.

After substitution of all these approximations in the results derived from (A7) and (A9) using the respective expressions (A8) and (A10) we find, after simplification, that

$$\begin{aligned} 12(C_0 + C_0^*) = 8\phi_0 &- \{4 - 2hu_0 - \tfrac{1}{2}h^3v_0(\partial u/\partial y)_0 + \tfrac{1}{2}h^3(\partial^2 u/\partial y^2)_0\}\,\phi_1 \\ &- \{4 - 2hv_0 - \tfrac{1}{2}h^3u_0(\partial v/\partial x)_0 + \tfrac{1}{2}h^3(\partial^2 v/\partial x^2)_0\}\,\phi_2 \\ &- \{4 + 2hu_0 + \tfrac{1}{2}h^3v_0(\partial u/\partial y)_0 - \tfrac{1}{2}h^3(\partial^2 u/\partial y^2)_0\}\,\phi_3 \\ &- \{4 + 2hv_0 + \tfrac{1}{2}h^3u_0(\partial v/\partial x)_0 - \tfrac{1}{2}h^3(\partial^2 v/\partial x^2)_0\}\,\phi_4 \\ &+ \sum_{n=5}^{8} d_n\phi_n, \end{aligned} \quad \text{(A15)}$$

where the coefficients $d_5$–$d_8$ are exactly those defined in (33). The remaining coefficients in (33) arise after simplification of the expression obtained when $C_0 + C_0^*$ defined by (A15) is substituted in the equation found by adding the left-hand sides of (16) and (31) and equating to zero. It is clear that the coefficients $d_5$–$d_8$ arise only from $C_0 + C_0^*$.

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